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The primary objective was to understand fundamental properties of stochastic partial differential equations. The main results obtained concern properties of level sets of the solution of the one-dimensional wave equation, and regularity properties of the two-dimensional wave equation driven by non-white Gaussian noise. Additional results were obtained in the areas of stochastic optimization and stability of random matrix models.

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Stochastic Hyperbolic and Parabolic Partial Differential Equations

FINAL REPORT

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Statement of the problem studied

The central problem was to understand fundamental properties of stochastic partial differential equations. These included properties of level sets of the solutions of the one-dimensional dimensional wave equation, regularity properties of solutions of the two-dimensional wave equation driven by non-white Gaussian noise, and properties of parabolic equations. These equations describe wave or heat propagation in random media with random sources.

Additional problems examined were stochastic optimization problems that involve switching between diffusions or random walks, and the use of random matrices as models for studying the stability of large ecosystems.

Summary of results

THE ONE-DIMENSIONAL WAVE EQUATION. After a canonical transformation, the solution of this equation is given by a process indexed by the nonnegative orthant known as the Brownian sheet. Significant effort was devoted to the study of level sets of this process. The main results, obtained jointly with John B. Walsh, concern the local geometric structure of the level sets $\{W = \alpha\}$, $\alpha > 0$, where $W = \{W(s,t),\ s \geq 0,\ t \geq 0\}$ is a standard Brownian sheet. More precisely, the focus has been on "mapping out" this level set on a microscopic scale in the neighborhood of particular point: the first hit (S, t_0) of the level set along a horizontal line.

We established that the point (S, t_0) is not on the boundary of any connected component of $\{W > \alpha\}$. Furthermore, it can be surrounded entirely by a curve in $\{W < \alpha\}$. Questions concerning the maximal thorn-shaped neighborhood that can be fit around the segment $]0, S[\, \times \{t_0\}$ while still avoiding $\{W > \alpha\}$ have been completely answered by an integral test concerning the function τ which defines the boundary of the thorn T_τ, where

$$T_\tau = \{(s,t) : 1 \leq t \leq \tau(S-s),\ 0 < s \leq S\}.$$

Main result. Assuming only that $s \mapsto \tau(s)/s$ is increasing, we have shown that T_τ is *initially* contained in $\{W < \alpha\}$ if and only if

$$I(\tau) = \int_{0+} \left(\frac{\tau(s)}{s}\right)^{\frac{1}{2}} \frac{ds}{s} < \infty .$$

In the special case where

$$\tau(s) = s \left(\log \frac{1}{s}\right)^{-2} \left(\log\log \frac{1}{s}\right)^{-\beta},$$

the integral $I(\tau)$ is finite if and only if $\beta > 2$.

Many further results concerning the size, shape and spacing of components of $\{W > \alpha\}$ in the neighborhood of (S, t_0) have also been obtained: see [8].

Another direction of study concerned excursions of the Brownian sheet. The paper [9] studies the restriction of W to a single connected component of $\{W > \alpha\}$. This is analogous in spirit to studying an excursion of Brownian motion. Here, geometric properties of the component are of particular interest. In the neighborhood of (S, t_0), the Brownian sheet can be written as

$$W(S-u, 1+v/S) = 1 + B(v) - b(s) - x(u, v/S), \tag{1}$$

where $B = \{B(v),\ v \geq 0\}$ is a standard Brownian motion started at the origin, $b = \{b(u),\ u \geq 0\}$ is a Bessel(3) process also started at the origin and independent of B, and x is comparatively small. Therefore, it is natural to look at the set $\{B > b\} = \{(u,v) : B(v) > b(u)\}$, or equivalently, at excursions of the process $Y(u,v) = B(v) - b(u)$.

Main results. The following questions concerning the set $\{B > b\}$ have been answered: what is the probability that a point in the non-negative quadrant $\mathbb{R}^2_+$ belongs to a particular connected component of $\{B > b\}$, and what is the expected area of this component given the (maximal) height of the excursion of Y in this component. Explicit formulas for these quantities have been obtained. The formula for the expected area is of the following type:

$$E\{\text{ area of component } \mid \text{ height }\} = \sum_{n \in N} p_n,$$

where p_n is an integral, over a particular simplex in $7 + 6(n-1)$-dimensional Euclidean space, of functions f and g involving densities of functionals of two independent Bessel(3) processes. These functions are expressed in terms of several series whose general terms can be computed from the standard Gaussian density. Since these series converge rapidly, they can be evaluated numerically. This numerical computation has been carried out and agrees with results from direct simulations of the processes b and B.

Also addressed in [9] is the question of comparing the asymptotic distribution of the area of components of the Brownian sheet, suitably normalized, with the area of components of $\{B > b\}$. A precise convergence statement is formulated and established in this paper. It should be emphasized that this is the first result in the literature concerning the area of excursions of the Brownian sheet.

Further results. The results obtained above have motivated further ongoing work that R.C. Dalang is currently pursuing and will be completed in the months to come. In particular, in joint work with T. Mountford [6], the PI has shown that a curve in the level set of the Brownian sheet must be nowhere-differentiable. This is the appropriate analogue of a result of nowhere-differentiability of Brownian motion due to Paley, Wiener and Zygmund [19]. On the other hand, Dvoretzky, Erdös and Kakutani [10] proved a very fine result concerning the nonexistence of points of increase of Brownian motion. In [5], T. Mountford and R.C. Dalang have shown that the analogous statement for the Brownian sheet is false, namely, with probability one, there exist monotone curves along which the Brownian sheet has points of increase.

THE TWO-DIMENSIONAL WAVE EQUATION. Motivated by applications in atmospheric science in which some models use random terms other than space-time white noise, the two PI's R.C. Dalang and N. Frangos have studied the wave equation

$$u_{tt} - \Delta u = f(u)\, G(dt, dx), \tag{2}$$

where G is a Gaussian random field with covariance given by $\delta(t-s)R(|x-y|)$, and f is a bounded Lipschitz function. If R is bounded, then it is well known [18] that (2) has a continuous solution, but if $R(|x|)$ is unbounded in the neighborhood of the origin, then the situation is much more delicate. While the linear form of (2) always has a solution in the space of generalized processes, the questions arises as to what conditions $R(|x|)$ must satisfy in order that the solution exist in the space of real-valued stochastic processes. The PI's have shown that (2) has a process-solution if and only if

$$\int_{0+} u(\ln 1/u) R(u)\, du < \infty.$$

However, this stochastic process need not be continuous. Under the stronger condition

$$\int_{0+} u^{1-\epsilon} R(u)\, du < \infty \qquad \text{for some } \varepsilon > 0,$$

the PI's have shown in [4] that the process solution of (2) has a continuous version.

The case where G is a Lévy random measure is also of interest. In this case, R.C. Dalang and graduate student Hou Qiang have shown for a natural class of Lévy processes that for large class of domains, the solution to (2) has the germ-field Markov property.

PARABOLIC EQUATIONS. Jointly with D. Nualart, R.C. Dalang has been studying the space-time Markov property of the equation

$$\frac{\partial}{\partial t} u(t,x) - \frac{\partial^2}{\partial x^2} u(t,x) = f(u(t,x)) \dot{W}(dt, dx).$$

While significant effort has been devoted to this difficult problem and much progress has been made, the main results are yet to be obtained and are the subject of continuing work.

STABILITY OF ECOSYSTEMS. Jointly with H.M. Hastings and M.A. Schreiber, N. Frangos has studied the stability of large ecosystems using random matrix models. The evolution of certain ecosystems can often be described by a parabolic partial differential equation of the form

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + \dot{W}(dt, dx).$$

Through discretization, this equation leads a system of stochastic differential equations with large random matrices and noise terms. A relationship between Lyapounov stability and low variability of the ecosystem has been established.

STOCHASTIC OPTIMIZATION. Research in this area has progressed on two fronts, related to the monograph [2] and to the papers [1, 3], jointly with R. Cairoli. The papers [1, 3] contain a complete solution to the following optimal switching problem, first studied by Mandelbaum [16] and Mandelbaum, Shepp and Vanderbei [17] under certain regularity assumptions. Consider two independent Brownian motions $X^1 = (X^1_{t^1}, t^1 \in \mathbb{R}_+)$ and $X^2 = (X^2_{t^2}, t^2 \in \mathbb{R}_+)$ killed at the endpoints of the intervals $[0, N^1]$ and $[0, N^2]$ respectively, where N^1 and N^2 are positive numbers. Imagine that an observer can control the evolution of X^1 and X^2 separately, that is, can leave t^1 fixed and let t^2 increase or leave t^2 fixed and let t^1 increase. The observer can switch from one direction to the other at any time. This determines a process $((X^1_{t^1}, X^2_{t^2}), (t^1, t^2) \in \mathbb{R}^2_+)$ which evolves in $D = [0, N^1] \times [0, N^2]$. Assume that the observer can choose the switching strategy and the time at which the evolution ends, knowing that at that time he will receive a reward which only depends on the state of the process. This reward is represented by a non-negative real-valued payoff function f defined on D which vanishes in the interior of D, and the objective is to maximize the expected reward. The problem has a discrete-time analogue, in which the Brownian motions are replaced by random walks.

Main result. The main result, obtained jointly by R. Cairoli and the proposer, is a complete description of the solution to this optimal switching problem both in discrete and in continuous time, under the minimal "almost necessary" regularity assumption on the data, namely, no assumption in the discrete case and "f is upper-semicontinuous" in the continuous case. This can be compared with the assumption of [17], namely that the boundary data is twice continuously differentiable and strictly concave. The methods used by R. Cairoli and the proposer are significantly different from those in this reference, which relied on the so-called "Principle of Smooth Fit". In the general setting, this principle does not apply, since when the boundary data is not regular, the "fit is not smooth". We proposed a combinatorial solution in the discrete case, and a limiting argument to extend this to the continuous case. The regions where the vertical (resp. horizontal) control is optimal are described explicitly in terms of the boundary data.

The monograph "Sequential stochastic optimization". The monograph [2] presents a unified mathematical theory of sequential stochastic optimization, with emphasis on the problems of optimal stopping and control of stochastic processes in the presence of incomplete information, together with several applications, including sequential statistical testing involving several populations and the multi-armed bandit problem. Much of the material presented here is either new or appears in a book for the first time.

The monograph demonstrates how the theory of multiparameter processes has proved to be an excellent tool for formulating and solving optimal stopping and sequential control problems, as noticed by Mandelbaum [15] and pursued by El Karoui and Karatzas [11, 12]. In sequential problems, where decisions are made in discrete time, processes indexed by the integer lattice or other discrete partially ordered sets provide the framework within which these problems can be given a rigorous mathematical formulation.

In contexts in which there are only finitely many states of nature, the optimization problem usually reduces to a problem in combinatorial optimization, in which graph-theoretic and integer programming methods are important (as in the paper [7]). The discrete case on a general probability space covers many sampling problems, the multi-armed bandit problem and stochastic scheduling problems.

New results contained in the monograph include: an extension to arrays of exchangeable random variables of a result of Krengel and Sucheston [13] concerning linear embedding of arrays of independent variables; an extension of a criterion for accessibility of stopping points in the plane to stopping points in $\mathbb{N}^d$, $d > 1$; and a characterization of three-dimensional filtrations with respect to which all stopping points are accessible. A very general control problem, sufficient to cover applications in which information and costs depend on all prior actions and states of the system is formulated and solved. A characterization of Markov controls is provided, correcting a result of Lawler and Vanderbei [14]. And a detailed presentation of the optimal switching problem in discrete time mentioned above is provided.

List of Publications

1. R.C. Dalang and J.B. Walsh, "Geography of the Level Sets of the Brownian sheet", *Probability Theory and Related Fields* 96 (1993), 153-176.

2. R.C. Dalang and J.B. Walsh, "The Structure of a Brownian Bubble", *Probability Theory and Related Fields* 96 (1993), 475-501.

3. R.C. Dalang and J.B. Walsh, "Local Structure of Level Sets of the Brownian Sheet", Proceedings of the Bar-Ilan Symposium "Brownian Sheets: Recent Developments" (Sept. 1993), to appear.

4. R. Cairoli and R.C. Dalang, "Optimal switching between two Brownian motions", Proceedings of the American Mathematical Society Summer Research Institute in Stochastic Analysis at Cornell University (July 1993), to appear.

5. R. Cairoli and R.C. Dalang, "Sequential Stochastic Optimization" (monograph with 10 chapters, 350 pages, to be published in late 1994 by J. Wiley & Sons, Inc.).

6. N. Frangos, H.M. Hastings and M.A. Schreiber, "Stability of Structured Random Matrices", Proceedings of the Fourth International Colloquium on Differential Equations (Plovdiv, Bulgaria, 1993) (to appear).

7. N. Frangos and P. Imkeller, "Existence and Continuity of the Quadratic Variation of Strong Martingales", Proceedings of the 1993 conference on Convergence theorems in Probability and Ergodic Theory (to appear).

8. R. Cairoli and R.C. Dalang, Optimal Switching Between Two Random Walks, submitted to the *Annals of Probability* (28 pages).

9. R.C. Dalang and T. Mountford, Points of Increase of the Brownian Sheet (in preparation).

10. R.C. Dalang and T. Mountford, Nowhere-differentiablity of the Brownian sheet (in preparation).

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R. Dalang and N. Frangos

Report of inventions

None.

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[5] Dalang, R.C. & Mountford, T., Points of increase of the Brownian sheet (in preparation).

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